CS 6316 Machine Learning

Support Vector Machines and Kernel Methods

Yangfeng Ji

Information and Language Processing Lab
Department of Computer Science
University of Virginia

Overview

1. Review: Linear Functions
2. Separable Cases
3. Constrained Optimization
4. Non-separable Cases
5. Dual Optimization Problem
6. Kernel Methods

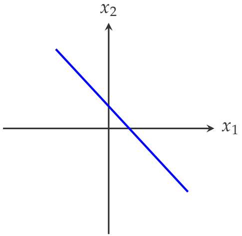
Readings: [Shalev-Shwartz and Ben-David, 2014, Chapter 15 & 16]

Review: Linear Functions

Linear Functions

Consider a two-dimensional case with $\boldsymbol{w} = (1, 1, -0.5)$

$$f(\boldsymbol{x}) = \boldsymbol{w}^\top \boldsymbol{x} + b = x_1 + x_2 - 0.5 \tag{1}$$



Different values of $f(\boldsymbol{x})$ map to different areas on this 2-D space. For example, the following equation defines the blue line L.

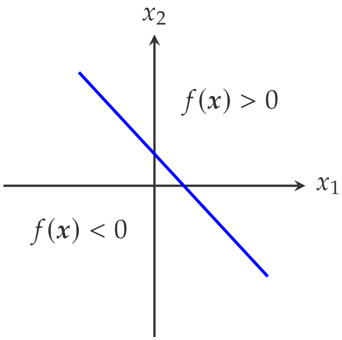
$$f(\boldsymbol{x}) = \boldsymbol{w}^\top \boldsymbol{x} + b = 0 \tag{2}$$

Properties of Linear Functions (Cont.)

Furthermore,

$$f(\boldsymbol{x}) = x_1 + x_2 - 0.5 = 0 \quad (3)$$

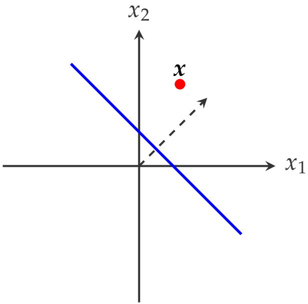
separates the 2-D space $\mathbb{R}^2$ into two half spaces



Properties of Linear Functions (Cont.)

The distance of point $\boldsymbol{x}$ to line $L : f(\boldsymbol{x}) = \langle \boldsymbol{w}, \boldsymbol{x} \rangle + b = 0$ is given by

$$\frac{f(\boldsymbol{x})}{\|\boldsymbol{w}\|_2} = \frac{\langle \boldsymbol{w}, \boldsymbol{x} \rangle + b}{\|\boldsymbol{x}\|_2} = \langle \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_2}, \boldsymbol{x} \rangle + \frac{b}{\|\boldsymbol{w}\|_2} \tag{4}$$

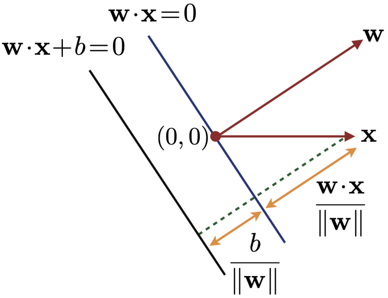


Separable Cases

Geometric Margin

The geometric margin of a linear binary classifier $h(x) = \langle w, x \rangle + b$ at a point x is its distance to the hyper-plane $\langle w, x \rangle = 0$

$$\rho_h(x) = \frac{|\langle w, x \rangle + b|}{\|w\|_2} \tag{5}$$



Geometric Margin (II)

The geometric margin of $h(\boldsymbol{x})$ on a set of examples $T = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}$ is the minimal distance over these examples

$$\rho_h(T) = \min_{\boldsymbol{x}' \in T} \rho_h(\boldsymbol{x}') \tag{6}$$

[Mohri et al., 2018, Page 80]

Half-Space Hypothesis Space

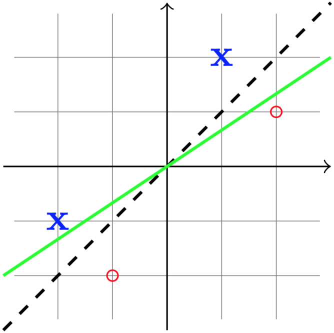
- Training set $S = \{(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_m, y_m)\}$ with $\boldsymbol{x}_i \in \mathbb{R}^d$ and $y_i \in \{+1, -1\}$
- If the training set is linearly separable

$$y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) > 0 \quad \forall i \in [m] \tag{7}$$

- Linearly separable cases
  - Existence of equation 7
  - All halfspace predictors that satisfy the condition in equation 7 are ERM hypotheses

Which Hypothesis is Better?

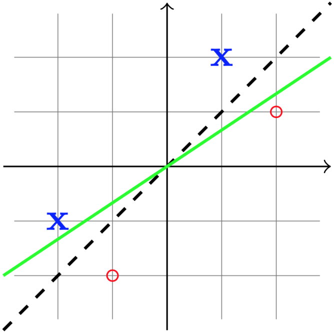
Is the one represented by the green line or the black dashed line?



[Shalev-Shwartz and Ben-David, 2014, Page 203]

Which Hypothesis is Better?

Is the one represented by the green line or the black dashed line?



- Intuitively, a hypothesis with larger *margin* is better, because it is more robust to noise
- Final definition of margin will be provided later

[Shalev-Shwartz and Ben-David, 2014, Page 203]

Hard SVM/Separable Cases

The mathematical formulation of the previous idea

$$\rho = \max_{(w,b)} \min_{i \in [m]} \frac{|\langle w, x_i \rangle + b|}{\|w\|_2} \tag{8}$$

$$\text{s.t. } y_i(\langle w, x_i \rangle + b) > 0 \quad \forall i \tag{9}$$

s.t. means *subject to* in optimization, to introduce constraints

Notations:

- $y_i(\langle w, x_i \rangle + b) > 0 \; \forall i$: guarantee (w, b) is an ERM hypothesis

Hard SVM/Separable Cases

The mathematical formulation of the previous idea

$$\rho = \max_{(\boldsymbol{w},b)} \min_{i\in[m]} \frac{|\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b|}{\|\boldsymbol{w}\|_2} \tag{8}$$

$$\text{s.t. } y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) > 0 \quad \forall i \tag{9}$$

s.t. means *subject to* in optimization, to introduce constraints

Notations:

- $y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) > 0 \; \forall i$: guarantee $(\boldsymbol{w}, b)$ is an ERM hypothesis
- $\min_{i\in[m]}$: calculate the margin between a hyper-plane and a set of examples

Hard SVM/Separable Cases

The mathematical formulation of the previous idea

$$\rho = \max_{(\boldsymbol{w},b)} \min_{i\in[m]} \frac{|\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b|}{\|\boldsymbol{w}\|_2} \quad (8)$$

$$\text{s.t. } y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b) > 0 \quad \forall i \quad (9)$$

s.t. means *subject to* in optimization, to introduce constraints

Notations:

- $y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b) > 0 \; \forall i$: guarantee $(\boldsymbol{w}, b)$ is an ERM hypothesis
- $\min_{i\in[m]}$: calculate the margin between a hyper-plane and a set of examples
- $\max_{(\boldsymbol{w},b)}$: maximize the margin

Hard SVM/Separable Cases

The mathematical formulation of the previous idea

$$\rho = \max_{(\boldsymbol{w},b)} \min_{i\in[m]} \frac{|\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b|}{\|\boldsymbol{w}\|_2} \quad (8)$$

$$\text{s.t. } y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) > 0 \quad \forall i \quad (9)$$

s.t. means *subject to* in optimization, to introduce constraints

Notations:

- $y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) > 0 \; \forall i$: guarantee $(\boldsymbol{w}, b)$ is an ERM hypothesis
- $\min_{i\in[m]}$: calculate the margin between a hyper-plane and a set of examples
- $\max_{(\boldsymbol{w},b)}$: maximize the margin

Overall, the optimization problem is to find a hypothesis that (1) classifies all training example correctly and (2) also has the largest margin.

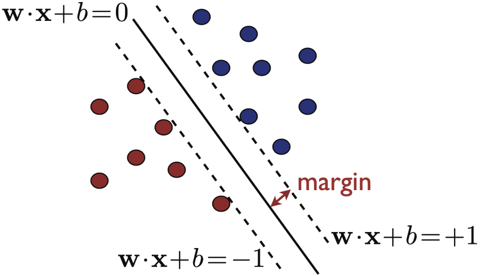
Illustration

Original form

$$\rho = \max_{(w,b)} \min_{i\in[m]} \frac{|\langle w, x_i\rangle + b|}{\|w\|_2} \quad (10)$$

$$\text{s.t. } y_i(\langle w, x_i\rangle + b) > 0 \quad \forall i \quad (11)$$

An example with the margin as 1



Alternative Forms

- Original form

$$\rho = \max_{(\boldsymbol{w},b)} \min_{i\in[m]} \frac{|\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b|}{\|\boldsymbol{w}\|_2} \quad (12)$$

$$\text{s.t. } y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b) > 0 \quad \forall i \quad (13)$$

Alternative Forms

- Original form

$$\rho = \max_{(w,b)} \min_{i\in[m]} \frac{|\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b|}{\|\boldsymbol{w}\|_2} \tag{12}$$

$$\text{s.t. } y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b) > 0 \quad \forall i \tag{13}$$

- Alternative form 1

$$\rho = \max_{(w,b)} \min_{i\in[m]} \frac{y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b)}{\|\boldsymbol{w}\|_2} \tag{14}$$

Alternative Forms

- Original form

$$\rho = \max_{(\boldsymbol{w},b)} \min_{i\in[m]} \frac{|\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b|}{\|\boldsymbol{w}\|_2} \tag{12}$$

$$\text{s.t. } y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b) > 0 \quad \forall i \tag{13}$$

- Alternative form 1

$$\rho = \max_{(\boldsymbol{w},b)} \min_{i\in[m]} \frac{y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b)}{\|\boldsymbol{w}\|_2} \tag{14}$$

- Alternative form 2

$$\rho = \max_{(\boldsymbol{w},b):\ \min_{i\in[m]} y_i(\langle \boldsymbol{w},\boldsymbol{x}_i\rangle + b = 1} \frac{1}{\|\boldsymbol{w}\|_2} \tag{15}$$

$$= \max_{(\boldsymbol{w},b):\ y_i(\langle \boldsymbol{w},\boldsymbol{x}_i\rangle + b \geq 1} \frac{1}{\|\boldsymbol{w}\|_2} \tag{16}$$

Alternative Forms (II)

- Alternative form 2

$$\rho = \max_{(\boldsymbol{w},b):\ y_i(\langle \boldsymbol{w},\boldsymbol{x}_i\rangle + b \geq 1} \frac{1}{\|\boldsymbol{w}\|_2} \tag{17}$$

- Alternative form 3: Quadratic programming (QP)

$$\begin{aligned} \min_{(\boldsymbol{w},b)} \ & \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\ \text{s.t.} \ & y_i(\langle \boldsymbol{w},\boldsymbol{x}_i\rangle + b) \geq 1, \quad \forall i \in [m] \end{aligned} \tag{18}$$

which is a constrained optimization problem that can be solved by standard QP packages

Alternative Forms (II)

- Alternative form 2

$$\rho = \max_{(\boldsymbol{w},b):\ y_i(\langle \boldsymbol{w},\boldsymbol{x}_i\rangle + b \geq 1} \frac{1}{\|\boldsymbol{w}\|_2} \tag{17}$$

- Alternative form 3: Quadratic programming (QP)

$$\begin{aligned} \min_{(\boldsymbol{w},b)} \quad & \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\ \text{s.t.} \quad & y_i(\langle \boldsymbol{w},\boldsymbol{x}_i\rangle + b) \geq 1, \quad \forall i \in [m] \end{aligned} \tag{18}$$

  which is a constrained optimization problem that can be solved by standard QP packages

- *Exercise*: Solve a SVM problem with quadratic programming

Unconstrained Optimization Problem

The quadratic programming problem with constraints can be converted to an unconstrained optimization problem with the Lagrangian method

$$L(\boldsymbol{w}, b, \boldsymbol{\alpha}) = \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_{i=1}^{m} \alpha_i (y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) - 1) \tag{19}$$

where

- $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$ is the Lagrange multiplier, and
- $\alpha_i \geq 0$ is associated with the i-th training example

Unconstrained Optimization Problem

The quadratic programming problem with constraints can be converted to an unconstrained optimization problem with the Lagrangian method

$$L(\boldsymbol{w}, b, \boldsymbol{\alpha}) = \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_{i=1}^{m} \alpha_i (y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) - 1) \tag{19}$$

where

- $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$ is the Lagrange multiplier, and
- $\alpha_i \geq 0$ is associated with the i-th training example

Can you identify the similarity between Eq. 19 and regularized linear regression?

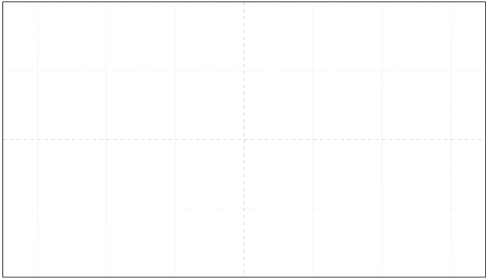
SVM Online Demo

Interactive demo of Support Vector Machines (SVM)

February 12, 2018

tags: c++ , machine-learning , svm , wasm

*Note: you may have to **disable your adblocker** for this demo to work.*



Toggle | Clear Points $\nu =$ 0.25

Kernel: Linear $\gamma =$ 1.0 $c_0 =$ 0.0

$$K(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$$

Link

Constrained Optimization

Constrained Optimization Problems: Definition

A generic formulation of constrained optimization

- $\mathcal{X} \subseteq \mathbb{R}^d$ and
- $f, g_i : \mathcal{X} \rightarrow \mathbb{R}, \forall i \in [m]$

Then, a constrained optimization problem is defined in the form of

$$\min_{\boldsymbol{x} \in \mathcal{X}} \quad f(\boldsymbol{x}) \tag{20}$$

$$\text{s.t.} \quad g_i(\boldsymbol{x}) \leq 0, \forall i \in [m] \tag{21}$$

Constrained Optimization Problems: Definition

A generic formulation of constrained optimization

- $\mathcal{X} \subseteq \mathbb{R}^d$ and
- $f, g_i : \mathcal{X} \rightarrow \mathbb{R}, \forall i \in [m]$

Then, a constrained optimization problem is defined in the form of

$$\min_{x \in \mathcal{X}} \quad f(\boldsymbol{x}) \tag{20}$$

$$\text{s.t.} \quad g_i(\boldsymbol{x}) \leq 0, \forall i \in [m] \tag{21}$$

Comments

- Unlike a learning problem, here $\boldsymbol{x}$ is the target variable for optimization
- Special cases of $g_i(\boldsymbol{x})$: (1) $g_i(\boldsymbol{x}) = 0$, (2) $g_i(\boldsymbol{x}) \geq 0$, and (3) $g_i(\boldsymbol{x}) \leq b$

Lagrangian

The Lagrangian associated to the general constrained optimization problem defined in equation 20 – 21 is the function defined over $\mathcal{X} \times \mathbb{R}_+^m$ as

$$L(\boldsymbol{x}, \boldsymbol{\alpha}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \alpha_i g_i(\boldsymbol{x}) \tag{22}$$

where

- $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_+^m$
- $\alpha_i \geq 0$ for any $i \in [m]$

Karush-Kuhn-Tucker's Theorem

Assume that $f, g_i : \mathcal{X} \to \mathbb{R}, \forall i \in [m]$ are convex and differentiable and that the constraints are qualified. Then $\boldsymbol{x}'$ is a solution of the constrained problem if and only if there exist $\boldsymbol{\alpha}' \geq 0$ such that

$$\nabla_{\boldsymbol{x}} L(\boldsymbol{x}', \boldsymbol{\alpha}') = \nabla_{\boldsymbol{x}} f(\boldsymbol{x}') + \boldsymbol{\alpha}' \cdot \nabla_{\boldsymbol{x}} g(\boldsymbol{x}) = 0 \quad (23)$$

$$\nabla_{\boldsymbol{\alpha}} L(\boldsymbol{x}, \boldsymbol{\alpha}) = g(\boldsymbol{x}') \leq 0 \quad (24)$$

$$\boldsymbol{\alpha}' \cdot g(\boldsymbol{x}') = \sum_{i=1}^{m} \alpha'_i g_i(\boldsymbol{x}') = 0 \quad (25)$$

Equations 23 – 25 are called KKT conditions

[Mohri et al., 2018, Thm B.30]

KKT in SVM

Apply the KKT conditions to the SVM problem

$$L(\boldsymbol{w}, b, \boldsymbol{\alpha}) = \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_{i=1}^{m} \alpha_i (y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) - 1) \tag{26}$$

We have

$$\nabla_w L = \boldsymbol{w} - \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i = 0 \quad \Rightarrow \quad \boldsymbol{w} = \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i$$

KKT in SVM

Apply the KKT conditions to the SVM problem

$$L(\boldsymbol{w}, b, \boldsymbol{\alpha}) = \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_{i=1}^{m} \alpha_i (y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) - 1) \tag{26}$$

We have

$$\nabla_{w} L = \boldsymbol{w} - \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i = 0 \quad \Rightarrow \quad \boldsymbol{w} = \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i$$

$$\nabla_{b} L = -\sum_{i=1}^{m} \alpha_i y_i = 0 \quad \Rightarrow \quad \sum_{i=1}^{m} \alpha_i y_i = 0$$

KKT in SVM

Apply the KKT conditions to the SVM problem

$$L(\boldsymbol{w}, b, \boldsymbol{\alpha}) = \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_{i=1}^{m} \alpha_i (y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) - 1) \tag{26}$$

We have

$$\nabla_w L = \boldsymbol{w} - \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i = 0 \quad \Rightarrow \quad \boldsymbol{w} = \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i$$

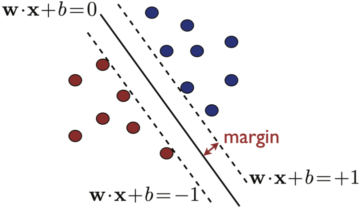
$$\nabla_b L = -\sum_{i=1}^{m} \alpha_i y_i = 0 \quad \Rightarrow \quad \sum_{i=1}^{m} \alpha_i y_i = 0$$

$$\forall i, \alpha_i (y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) - 1) = 0 \quad \Rightarrow \quad \alpha_i = 0 \text{ or } y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) = 1$$

Support Vectors

Consider the implication of the last equation in the previous page, $\forall i$

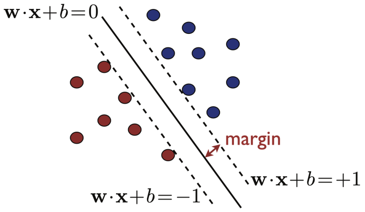
- $\alpha_i > 0$ and $y_i(\langle w, x_i \rangle + b) = 1$
  or



Support Vectors

Consider the implication of the last equation in the previous page, $\forall i$

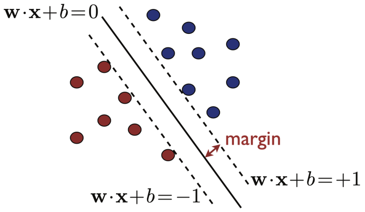
- $\alpha_i > 0$ and $y_i(\langle w, x_i \rangle + b) = 1$
  or
- $\alpha_i = 0$ and $y_i(\langle w, x_i \rangle + b) \geq 1$



Support Vectors

Consider the implication of the last equation in the previous page, $\forall i$

- $\alpha_i > 0$ and $y_i(\langle w, x_i \rangle + b) = 1$
  or
- $\alpha_i = 0$ and $y_i(\langle w, x_i \rangle + b) \geq 1$



$$w = \sum_{i=1}^{m} \alpha_i y_i x_i \tag{27}$$

- Examples with $\alpha_i > 0$ are called **support vectors**
- In $\mathbb{R}^d$, $d + 1$ examples are sufficient to define a hyper-plane

Non-separable Cases

Non-separable Cases

Recall the separable case:

$$\begin{aligned} &\min_{(\boldsymbol{w},b)} \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\ &\text{s.t. } y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) \geq 1, \quad \forall i \in [m] \end{aligned} \tag{28}$$

Non-separable Cases

Recall the separable case:

$$\begin{aligned} \min_{(\boldsymbol{w},b)} \ & \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\ \text{s.t. } & y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) \geq 1, \quad \forall i \in [m] \end{aligned} \tag{28}$$

For non-separable cases, there always exists an $\boldsymbol{x}_i$, such that

$$y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) \ngeq 1 \tag{29}$$

or, we can formulate it as

$$y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) \geq 1 - \xi_i \tag{30}$$

with $\xi_i \geq 0$

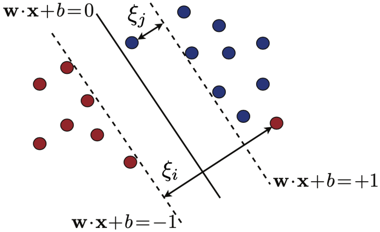
Geometric Meaning of ξ_i

Consider the relaxed constraint

$$y_i(\langle \boldsymbol{w}, x_i \rangle + b) \geq 1 - \xi_i \tag{31}$$

and three cases of ξ_i

- $\xi_i = 0$
- $0 < \xi_i < 1$
- $\xi_i \geq 1$



Non-separable Cases (II)

In general, the SVM problem of non-separable cases can be formulated as

$$\begin{aligned} \min_{(\boldsymbol{w},b)} \ & \frac{1}{2}\|\boldsymbol{w}\|_2^2 + C\sum_{i=1}^{m}\xi_i^p \\ \text{s.t. } & y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b) \geq 1 - \xi_i, \quad \forall i \in [m] \\ & \xi_i \geq 0 \end{aligned} \tag{32}$$

where $C \geq 0$, $p \geq 1$, and $\{\xi_i\}_{i=1}^m \geq 0$ are known as **slack variables** and are commonly used in optimization to define relaxed versions of constraints.

Lagrangian

Follows the same procedure as the separable cases, the Lagrangian is defined as

$$
\begin{aligned}
L(\boldsymbol{w}, b, \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = & \frac{1}{2}\|\boldsymbol{w}\|_2^2 + C\sum_{i=1}^{m}\xi_i \\
& - \sum_{i=1}^{m}\alpha_i(y_i(\boldsymbol{w}^\top\boldsymbol{x}_i + b) - 1 + \xi_i) \\
& - \sum_{i=1}^{m}\beta_i\xi_i
\end{aligned} \tag{33}
$$

with $\alpha_i, \beta_i \geq 0$

Lagrangian

Follows the same procedure as the separable cases, the Lagrangian is defined as

$$
\begin{aligned}
L(\boldsymbol{w}, b, \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = &\frac{1}{2}\|\boldsymbol{w}\|_2^2 + C\sum_{i=1}^{m}\xi_i \\
&- \sum_{i=1}^{m}\alpha_i(y_i(\boldsymbol{w}^\top \boldsymbol{x}_i + b) - 1 + \xi_i) \\
&- \sum_{i=1}^{m}\beta_i\xi_i
\end{aligned}
\tag{33}
$$

with $\alpha_i, \beta_i \geq 0$

Exercise: show the KKT conditions of equation 33

Support Vectors

The first two equations in the KKT conditions are similar to the separable cases, and the rest are

$$\alpha_i + \beta_i = C \tag{34}$$

$$\alpha_i = 0 \quad \text{or} \quad y_i(\boldsymbol{w}^\top \boldsymbol{x}_i + b) = 1 - \xi_i \tag{35}$$

$$\beta_i = 0 \quad \text{or} \quad \xi_i = 0 \tag{36}$$

Depending the value of ξ_i, there are two types of support vectors

- $\xi_i = 0$: $\beta_i \geq 0$ and $0 < \alpha_i \leq C$
    - $\boldsymbol{x}_i$ may lie on the marginal hyper-planes (as in the separable case)

Support Vectors

The first two equations in the KKT conditions are similar to the separable cases, and the rest are

$$\alpha_i + \beta_i = C \tag{34}$$

$$\alpha_i = 0 \quad \text{or} \quad y_i(\boldsymbol{w}^\top \boldsymbol{x}_i + b) = 1 - \xi_i \tag{35}$$

$$\beta_i = 0 \quad \text{or} \quad \xi_i = 0 \tag{36}$$

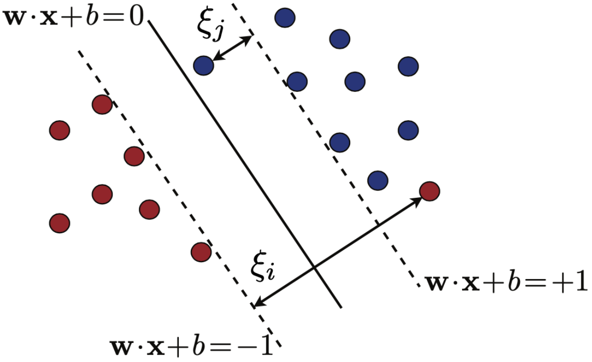
Depending the value of ξ_i, there are two types of support vectors

- $\xi_i = 0$: $\beta_i \geq 0$ and $0 < \alpha_i \leq C$
  - $\boldsymbol{x}_i$ may lie on the marginal hyper-planes (as in the separable case)
- $\xi_i > 0$: $\beta_i = 0$ and $\alpha_i = C$
  - $\boldsymbol{x}_i$ is an outlier

Support Vectors (II)

Two types of support vectors

- $\alpha_i = C$: $\boldsymbol{x}_i$ is an outlier
- $0 < \alpha_i < C$: $\boldsymbol{x}_i$ lies on the marginal hyper-planes



Dual Optimization Problem

Lagrangian

Combine the Lagrangian

$$
\begin{aligned}
L &= \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_{i=1}^{m} \alpha_i [y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) - 1] \\
&= \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_{i=1}^{m} \alpha_i y_i \langle \boldsymbol{w}, \boldsymbol{x}_i \rangle - b \sum_{i=1}^{m} \alpha_i y_i + \sum_{i=1}^{m} \alpha_i
\end{aligned}
$$

Lagrangian

Combine the Lagrangian

$$
\begin{aligned}
L &= \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_{i=1}^{m} \alpha_i [y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b) - 1] \\
&= \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_{i=1}^{m} \alpha_i y_i \langle \boldsymbol{w}, \boldsymbol{x}_i \rangle - b \sum_{i=1}^{m} \alpha_i y_i + \sum_{i=1}^{m} \alpha_i
\end{aligned}
$$

with some of the KKT conditions

$$
\boldsymbol{w} = \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i \tag{37}
$$

$$
\sum_{i=1}^{m} \alpha_i y_i = 0, \tag{38}
$$

we have ...

Dual Problem

$$L = \frac{1}{2} \| \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i \|_2^2 - \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y_i y_j \langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle - b \underbrace{\sum_{i=1}^{m} \alpha_i y_i}_{=0} + \sum_{i=1}^{m} \alpha_i \tag{39}$$

Dual Problem

$$L = \frac{1}{2}\|\sum_{i=1}^{m}\alpha_i y_i \boldsymbol{x}_i\|_2^2 - \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j\langle \boldsymbol{x}_i, \boldsymbol{x}_j\rangle - b\underbrace{\sum_{i=1}^{m}\alpha_i y_i}_{=0} + \sum_{i=1}^{m}\alpha_i \tag{39}$$

Given $\|\sum_{i=1}^{m}\alpha_i y_i \boldsymbol{x}_i\|_2^2 = \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j\langle \boldsymbol{x}_i, \boldsymbol{x}_j\rangle$, we have

$$L = -\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j\langle \boldsymbol{x}_i, \boldsymbol{x}_j\rangle + \sum_{i=1}^{m}\alpha_i \tag{40}$$

Dual Problem (II)

The dual optimization problem for SVMs of the separable cases is

$$\max_{\boldsymbol{\alpha}} \quad \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j \langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle \tag{41}$$

$$\text{s.t.} \quad \alpha_i \geq 0 \tag{42}$$

$$\sum_{i=1}^{m} \alpha_i y_i = 0 \ \forall i \in [m] \tag{43}$$

Dual Problem (II)

The dual optimization problem for SVMs of the separable cases is

$$\max_{\boldsymbol{\alpha}} \quad \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j \langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle \tag{41}$$

$$\text{s.t.} \quad \alpha_i \geq 0 \tag{42}$$

$$\sum_{i=1}^{m} \alpha_i y_i = 0 \ \forall i \in [m] \tag{43}$$

- Lagrange multiplier $\boldsymbol{\alpha}$ is also called dual variable
- This is an optimization problem only about $\boldsymbol{\alpha}$

Dual Problem (II)

The dual optimization problem for SVMs of the separable cases is

$$\max_{\boldsymbol{\alpha}} \quad \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j \langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle \tag{41}$$

$$\text{s.t.} \quad \alpha_i \geq 0 \tag{42}$$

$$\sum_{i=1}^{m} \alpha_i y_i = 0 \ \forall i \in [m] \tag{43}$$

- Lagrange multiplier $\boldsymbol{\alpha}$ is also called dual variable
- This is an optimization problem only about $\boldsymbol{\alpha}$
- The dual problem is defined on the inner product $\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle$

Primal and Dual Problem

- Primal problem

$$\begin{aligned} \min_{(\boldsymbol{w},b)} \quad & \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\ \text{s.t.} \quad & y_i(\langle \boldsymbol{w}, \boldsymbol{x}_i\rangle + b) \geq 1, \quad \forall i \in [m] \end{aligned} \tag{44}$$

- Dual problem

$$\begin{aligned} \max_{\boldsymbol{\alpha}} \quad & \sum_{i=1}^{m} \alpha_i - \frac{1}{2}\sum_{i,j=1}^{m} \alpha_i\alpha_j y_i y_j \langle \boldsymbol{x}_i, \boldsymbol{x}_j\rangle \\ \text{s.t.} \quad & \sum_{i=1}^{m} \alpha_i y_i = 0 \text{ and } \alpha_i \geq 0 \ \forall i \in [m] \end{aligned} \tag{45}$$

- These two problems are equivalent

[Boyd and Vandenberghe, 2004, Chapter 5]

SVM Hypothesis, revisited

Once we solve the dual problem with $\boldsymbol{\alpha}$, we have the solution of $\boldsymbol{w}$ as

$$\boldsymbol{w} = \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i \tag{46}$$

and the hypothesis $h(\boldsymbol{x})$ as

$$h(\boldsymbol{x}) = \text{sign}(\langle \boldsymbol{w}, \boldsymbol{x} \rangle + b) \tag{47}$$

(49)

SVM Hypothesis, revisited

Once we solve the dual problem with $\boldsymbol{\alpha}$, we have the solution of $\boldsymbol{w}$ as

$$\boldsymbol{w} = \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i \tag{46}$$

and the hypothesis $h(\boldsymbol{x})$ as

$$\begin{aligned} h(\boldsymbol{x}) &= \text{sign}(\langle \boldsymbol{w}, \boldsymbol{x} \rangle + b) && (47) \\ &= \text{sign}(\langle \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i, \boldsymbol{x} \rangle + b) && (48) \\ & && (49) \end{aligned}$$

SVM Hypothesis, revisited

Once we solve the dual problem with $\boldsymbol{\alpha}$, we have the solution of $\boldsymbol{w}$ as

$$\boldsymbol{w} = \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i \tag{46}$$

and the hypothesis $h(\boldsymbol{x})$ as

$$\begin{aligned}
h(\boldsymbol{x}) &= \text{sign}(\langle \boldsymbol{w}, \boldsymbol{x} \rangle + b) & (47)\\
&= \text{sign}(\langle \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i, \boldsymbol{x} \rangle + b) & (48)\\
&= \text{sign}(\sum_{i=1}^{m} \alpha_i y_i \langle \boldsymbol{x}_i, \boldsymbol{x} \rangle + b) & (49)
\end{aligned}$$

SVM Hypothesis, revisited

Once we solve the dual problem with $\boldsymbol{\alpha}$, we have the solution of $\boldsymbol{w}$ as

$$\boldsymbol{w} = \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i \tag{46}$$

and the hypothesis $h(\boldsymbol{x})$ as

$$\begin{aligned} h(\boldsymbol{x}) &= \operatorname{sign}(\langle \boldsymbol{w}, \boldsymbol{x} \rangle + b) & (47)\\ &= \operatorname{sign}(\langle \sum_{i=1}^{m} \alpha_i y_i \boldsymbol{x}_i, \boldsymbol{x} \rangle + b) & (48)\\ &= \operatorname{sign}(\sum_{i=1}^{m} \alpha_i y_i \langle \boldsymbol{x}_i, \boldsymbol{x} \rangle + b) & (49) \end{aligned}$$

- In addition, we also have $b = y_i - \sum_{i=1}^{m} \alpha_i y_i \langle \boldsymbol{x}_i, \boldsymbol{x} \rangle$ for any $\boldsymbol{x}_i$ with $\alpha_i > 0$
- Therefore, everything can be represented in the form of dot product

Kernel Methods

Properties of Inner Product

In the solution of SVMs

$$
\begin{aligned}
h(\boldsymbol{x}) &= \text{sign}(\sum_{i=1}^{m} \alpha_i y_i \langle \boldsymbol{x}_i, \boldsymbol{x} \rangle + b) \\
b &= y_i - \sum_{i=1}^{m} \alpha_i y_i \langle \boldsymbol{x}_i, \boldsymbol{x} \rangle
\end{aligned}
\tag{50}
$$

Properties of Inner Product

In the solution of SVMs

$$
\begin{aligned}
h(\boldsymbol{x}) &= \text{sign}\Big(\sum_{i=1}^{m} \alpha_i y_i \langle \boldsymbol{x}_i, \boldsymbol{x} \rangle + b\Big) \\
b &= y_i - \sum_{i=1}^{m} \alpha_i y_i \langle \boldsymbol{x}_i, \boldsymbol{x} \rangle
\end{aligned}
\tag{50}
$$

Extend the capacity of SVMs by replacing the inner product $\langle \boldsymbol{x}_i, \boldsymbol{x} \rangle$ with a kernel function

$$
K(\boldsymbol{x}_i, \boldsymbol{x}) = \langle \Phi(\boldsymbol{x}_i), \Phi(\boldsymbol{x}) \rangle \tag{51}
$$

where $\Phi(\cdot)$ is a nonlinear mapping function.

SVMs with Kernel Functions

- Problem definition

$$\max_{\boldsymbol{\alpha}} \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j K(\boldsymbol{x}_i, \boldsymbol{x}_j)$$
$$\text{s.t. } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{m} \alpha_i y_i = 0, i \in [m] \quad (52)$$

SVMs with Kernel Functions

- Problem definition

$$\max_{\boldsymbol{\alpha}} \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j K(\boldsymbol{x}_i, \boldsymbol{x}_j)$$
$$\text{s.t. } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{m} \alpha_i y_i = 0, i \in [m] \tag{52}$$

- Solution: separable case

$$h(\boldsymbol{x}) = \text{sign}\left(\sum_{i=1}^{m} \alpha_i y_i K(\boldsymbol{x}_i, \boldsymbol{x}) + b\right) \tag{53}$$

with $b = y_i - \sum_{j=1}^{m} \alpha_j y_j K(\boldsymbol{x}_j, \boldsymbol{x}_i)$ for any $\boldsymbol{x}_i$ with $\alpha_i > 0$

Examples: Polynomial Kernels

For any constant $\gamma > 0, c \geq 0$, a **polynomial kernel** of degree $d \in \mathbb{N}$ is the kernel K defined over $\mathbb{R}^n$ by

$$K(\boldsymbol{x}, \boldsymbol{x}') = (\gamma \langle \boldsymbol{x}, \boldsymbol{x}' \rangle + c)^d, \forall \boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n \tag{54}$$

Examples: Polynomial Kernels

For any constant $\gamma > 0, c \geq 0$, a **polynomial kernel** of degree $d \in \mathbb{N}$ is the kernel K defined over $\mathbb{R}^n$ by

$$K(\boldsymbol{x}, \boldsymbol{x}') = (\gamma \langle \boldsymbol{x}, \boldsymbol{x}' \rangle + c)^d, \forall \boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n \tag{54}$$

Special cases

- $d = 1$: $K(\boldsymbol{x}, \boldsymbol{x}') = \gamma \langle \boldsymbol{x}, \boldsymbol{x}' \rangle + c$
- $d = 2$: $K(\boldsymbol{x}, \boldsymbol{x}') = (\gamma \langle \boldsymbol{x}, \boldsymbol{x}' \rangle + c)^2$

Examples: Polynomial Kernels (II)

For the special case with $d = 2$, assume $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^2$ (let $\gamma = 1$ for simplicity)

$$
\begin{aligned}
K(\boldsymbol{x}, \boldsymbol{x}') &= (\langle \boldsymbol{x}, \boldsymbol{x}' \rangle + c)^2 && (55) \\
&= (x_1 x_1' + x_2 x_2' + c)^2 && (56) \\
&= x_1^2 x_1'^2 + x_1 x_2 x_1' x_2' + c x_1 x_1' + x_1 x_2 x_1' x_2' \\
&\quad + x_2^2 x_2'^2 + c x_2 x_2' + c x_1 x_1' + c x_2 x_2' + c^2 && (57)
\end{aligned}
$$

Examples: Polynomial Kernels (II)

For the special case with $d = 2$, assume $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^2$ (let $\gamma = 1$ for simplicity)

$$
\begin{aligned}
K(\boldsymbol{x}, \boldsymbol{x}') &= (\langle \boldsymbol{x}, \boldsymbol{x}' \rangle + c)^2 && (55)\\
&= (x_1 x_1' + x_2 x_2' + c)^2 && (56)\\
&= x_1^2 x_1'^2 + x_1 x_2 x_1' x_2' + c x_1 x_1' + x_1 x_2 x_1' x_2' \\
&\quad + x_2^2 x_2'^2 + c x_2 x_2' + c x_1 x_1' + c x_2 x_2' + c^2 && (57)\\
&= x_1^2 x_1'^2 + x_2^2 x_2'^2 + 2 x_1 x'_1 x_2 x'_2 && (58)\\
&\quad + 2c x_1 x'_1 + 2c x_2 x'_2 + c^2 && (59)
\end{aligned}
$$

Examples: Polynomial Kernels (II)

For the special case with $d = 2$, assume $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^2$ (let $\gamma = 1$ for simplicity)

$$
\begin{aligned}
K(\boldsymbol{x}, \boldsymbol{x}') &= (\langle \boldsymbol{x}, \boldsymbol{x}' \rangle + c)^2 && (55)\\
&= (x_1 x_1' + x_2 x_2' + c)^2 && (56)\\
&= x_1^2 {x'}_1^2 + x_1 x_2 x_1' x_2' + c x_1 x_1' + x_1 x_2 x_1' x_2' \\
&\quad + x_2^2 {x'}_2^2 + c x_2 x_2' + c x_1 x_1' + c x_2 x_2' + c^2 && (57)\\
&= x_1^2 {x'}_1^2 + x_2^2 {x'}_2^2 + 2 x_1 {x'}_1 x_2 {x'}_2 && (58)\\
&\quad + 2c x_1 {x'}_1 + 2c x_2 {x'}_2 + c^2 && (59)\\
&= [x_1^2, x_2^2, \sqrt{2} x_1 x_2, \sqrt{2c} x_1, \sqrt{2c} x_2, c] \begin{bmatrix} {x'}_1^2 \\ {x'}_2^2 \\ \sqrt{2} {x'}_1 {x'}_2 \\ \sqrt{2c} {x'}_1 \\ \sqrt{2c} {x'}_2 \\ c \end{bmatrix}
\end{aligned}
$$

Examples: Polynomial Kernels (II)

For the special case with $d = 2$, assume $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^2$ (let $\gamma = 1$ for simplicity)

$$
\begin{aligned}
K(\boldsymbol{x}, \boldsymbol{x}') &= (\langle \boldsymbol{x}, \boldsymbol{x}' \rangle + c)^2 && (55)\\
&= (x_1 x_1' + x_2 x_2' + c)^2 && (56)\\
&= x_1^2 x'^2_1 + x_1 x_2 x_1' x_2' + c x_1 x_1' + x_1 x_2 x_1' x_2' \\
&\quad + x_2^2 x'^2_2 + c x_2 x_2' + c x_1 x_1' + c x_2 x_2' + c^2 && (57)\\
&= x_1^2 x'^2_1 + x_2^2 x'^2_2 + 2 x_1 x'_1 x_2 x'_2 && (58)\\
&\quad + 2c x_1 x'_1 + 2c x_2 x'_2 + c^2 && (59)\\
&= [x_1^2, x_2^2, \sqrt{2} x_1 x_2, \sqrt{2c} x_1, \sqrt{2c} x_2, c] \begin{bmatrix} x'^2_1 \\ x'^2_2 \\ \sqrt{2} x'_1 x'_2 \\ \sqrt{2c} x'_1 \\ \sqrt{2c} x'_2 \\ c \end{bmatrix}
\end{aligned}
$$

Exercise: Find out the $\Phi(\boldsymbol{x})$ function in $K(\boldsymbol{x}, \boldsymbol{x}') = (\langle \boldsymbol{x}, \boldsymbol{x}' \rangle + c)^3$

Examples: Polynomial Kernels (III)

Let $K(\boldsymbol{x}, \boldsymbol{x}') = \langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}') \rangle$, then

$$\Phi(\boldsymbol{x}) = [x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2c}x_1, \sqrt{2c}x_2, c]^\mathsf{T} \tag{60}$$

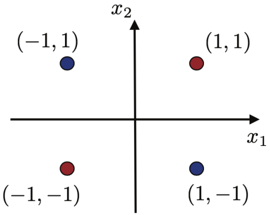
which maps a 2-D data point $\boldsymbol{x}$ into a 6-D space as $\Phi(\boldsymbol{x})$

Examples: Polynomial Kernels (III)

Let $K(\boldsymbol{x}, \boldsymbol{x}') = \langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}') \rangle$, then

$$\Phi(\boldsymbol{x}) = [x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2c}x_1, \sqrt{2c}x_2, c]^\mathsf{T} \tag{60}$$

which maps a 2-D data point $\boldsymbol{x}$ into a 6-D space as $\Phi(\boldsymbol{x})$ Recall the XOR problem

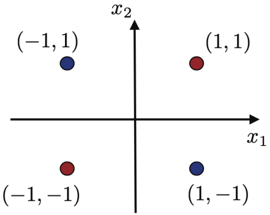


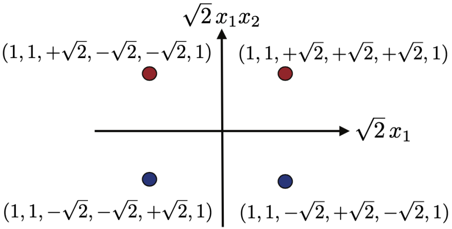
Examples: Polynomial Kernels (III)

Let $K(\boldsymbol{x}, \boldsymbol{x}') = \langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}') \rangle$, then

$$\Phi(\boldsymbol{x}) = [x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2c}x_1, \sqrt{2c}x_2, c]^\top \tag{60}$$

which maps a 2-D data point $\boldsymbol{x}$ into a 6-D space as $\Phi(\boldsymbol{x})$ Recall the XOR problem



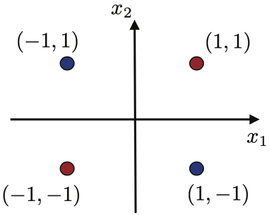


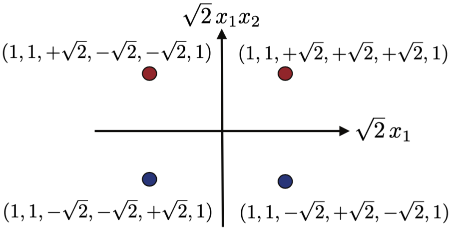
Examples: Polynomial Kernels (III)

Let $K(\boldsymbol{x}, \boldsymbol{x}') = \langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}') \rangle$, then

$$\Phi(\boldsymbol{x}) = [x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2c}x_1, \sqrt{2c}x_2, c]^\mathsf{T} \tag{60}$$

which maps a 2-D data point $\boldsymbol{x}$ into a 6-D space as $\Phi(\boldsymbol{x})$ Recall the XOR problem



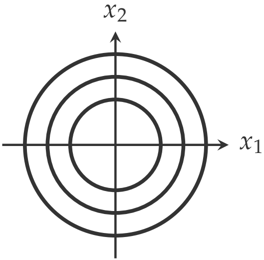


Try the online demo

Gaussian Kernels

For any constant $\gamma > 0$, a **Gaussian kernel** or **radial basis function** (RBF) is the kernel K defined over $\mathbb{R}^d$ by

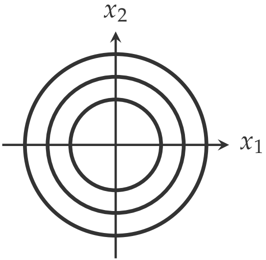
$$K(\boldsymbol{x}, \boldsymbol{x}') = \exp\left(-\gamma \|\boldsymbol{x}' - \boldsymbol{x}\|_2^2\right) \tag{61}$$



Gaussian Kernels

For any constant $\gamma > 0$, a **Gaussian kernel** or **radial basis function** (RBF) is the kernel K defined over $\mathbb{R}^d$ by

$$K(\boldsymbol{x}, \boldsymbol{x}') = \exp\left(-\gamma \|\boldsymbol{x}' - \boldsymbol{x}\|_2^2\right) \tag{61}$$



- What $\Phi(\boldsymbol{x})$ looks like in this case?
- What the effect of γ? (demo)

The Choice of Kernels

- The choice of $K(\boldsymbol{x}, \boldsymbol{x}')$ can be arbitrary, as long as the existence of $\Phi(\cdot)$ is guaranteed
  - For many cases, $\Phi(\cdot)$ cannot be found explicitly

[Mohri et al., 2018, Section 6.1 - 6.2]

The Choice of Kernels

- The choice of $K(\boldsymbol{x}, \boldsymbol{x}')$ can be arbitrary, as long as the existence of $\Phi(\cdot)$ is guaranteed
  - For many cases, $\Phi(\cdot)$ cannot be found explicitly
- Alternatively, we only need to make sure $K(\boldsymbol{x}, \boldsymbol{x}')$ is *positive definite symmetric* (PDS)
  - A kernel K is PDS if for any $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}$ the matrix $\mathbf{K}$ is symmetric positive semi-definite

$$\mathbf{K} = [K(\boldsymbol{x}_i, \boldsymbol{x}_j)]_{i,j} \in \mathbb{R}^{m \times m} \tag{62}$$

[Mohri et al., 2018, Section 6.1 - 6.2]

The Choice of Kernels

- The choice of $K(\boldsymbol{x}, \boldsymbol{x}')$ can be arbitrary, as long as the existence of $\Phi(\cdot)$ is guaranteed
  - For many cases, $\Phi(\cdot)$ cannot be found explicitly
- Alternatively, we only need to make sure $K(\boldsymbol{x}, \boldsymbol{x}')$ is *positive definite symmetric* (PDS)
  - A kernel K is PDS if for any $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\}$ the matrix $\mathbf{K}$ is symmetric positive semi-definite

$$\mathbf{K} = [K(\boldsymbol{x}_i, \boldsymbol{x}_j)]_{i,j} \in \mathbb{R}^{m \times m} \tag{62}$$

- A symmetric positive semi-definite matrix is defined as

$$\boldsymbol{c}^{\top} \mathbf{K} \boldsymbol{c} \geq 0 \tag{63}$$

[Mohri et al., 2018, Section 6.1 - 6.2]

Reference

Boyd, S. and Vandenberghe, L. (2004).
Convex optimization.
Cambridge university press.

Mohri, M., Rostamizadeh, A., and Talwalkar, A. (2018).
Foundations of machine learning.
MIT press.

Shalev-Shwartz, S. and Ben-David, S. (2014).
Understanding machine learning: From theory to algorithms.
Cambridge university press.